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On functional inequalities associated with Drygas functional equation

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Abstract

In the paper, the equivalence of the functional inequality

$$\|2f(x)+f(y)+f(-y)-f(x-y)\| \leq \|f(x+y)\| \quad (x, y \in G)$$

and the Drygas functional equation

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y) \quad (x, y \in G)$$

is proved for functions $f : G \to E$ where $(G, +)$ is an abelian group, $(E, < \cdot, \cdot >)$ is an inner product space, and the norm is derived from the inner product in the usual way.

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1 Introduction

Throughout the paper, $(G, +)$ will denote an abelian group and $(E, < \cdot, \cdot >)$ an inner product space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) with inner product $< \cdot, \cdot >$ and associated norm $\| \cdot \|$.
Gy. Maksa and P. Volkman proved in [13] the following

Theorem 1.1. *Let G be a group, E be an inner product space. If $f : G \to E$ be a function such that*

$$\|f(x)+f(y)\| \leq \|f(xy)\|$$

for all $x, y \in G$. Then f satisfies

$$f(xy) = f(x) + f(y)$$

for all $x, y \in G$.

In [9], A. Gilányi showed that if G is a 2-divisible abelian group, then the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\| \leq \|f(x+y)\| \ \ \textit{for all}\, x, y \in G \tag{1.1}$$

implies

$$f(x+y)+f(x-y)=2f(x)+2f(y) \ \ \textit{for all}\, x, y \in G, \tag{1.2}$$

and the commutativity of G may be replaced by the Kannappan condition
$f(xyz) = f(xzy)$, $x, y, z \in G$.
In [19] J. Rätz deleted the 2-divisibility of G, weakened the Kannappan condition and discussed

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variants of Gilányi result.
In [7], E. Elqorachi et al. proved that if the function $f : G \to E$ from G (an abelian 2-divisible group) to an inner product space E, satisfies the inequality

$$\|2f(x) + 2f(y) - f(x+\sigma(y))\| \leq \|f(x+y)\|, \quad x, y \in G,$$

where $\sigma : G \to G$ is an involution (i.e., $\sigma(x+y) = \sigma(x) + \sigma(y)$, $\sigma \circ \sigma(x) = \sigma(x)$ for all $x, y \in G$), then f satisfies the σ-quadratic functional equation

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y), \; x, y \in G.$$

The above-described effect: inequality implies equality was proved for some other functional equations. The interested reader can refer to [1], [2], [3], [4], [10], [11], [12], [14], [15], [16], [17], [18], [20] and [23] for a through account on the subject of functional inequalities.
We say that the function $f : G \to E$ satisfies the Drygas functional equation, if

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y) \tag{1.3}$$

for all $x, y \in G$.
The equation was introduced in [5], where the author was looking for characterizations of quasi inner product spaces, which in turn led to solutions of some problems in statistics and mathematical programming.
The functional equation (1.3) has been studied by Gy. Szabo [22], B. R. Ebanks et al. [6], V. A. Faǐziev and P. K. Sahoo [8]. The solutions of equation (1.3) in abelian group are obtained by H. Stetkær in [21].

The purpose of our paper is to show that if $f : G \to E$ satisfies the Drygas inequality

$$\|2f(x) + f(y) + f(-y) - f(x-y)\| \leq \|f(x+y)\| \;\; \textit{for all}\; x, y \in G,$$

then f satisfies the Drygas functional equation (1.3).

Throughout this paper, f^o and f^e denote the odd and even parts of f, respectively, i.e., $f^o(x) = \dfrac{f(x) - f(-x)}{2}$, $f^e(x) = \dfrac{f(x) + f(-x)}{2}$ for all $x \in G$.

2 Main result

Theorem 2.1. Let G be an abelian group, E be an inner product space and $f : G \to E$ be a mapping such that

$$\|2f(x) + f(y) + f(-y) - f(x-y))\| \leq \|f(x+y)\| \tag{1.4}$$

for all $x, y \in G$. Then f is a solution of the Drygas functional equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y), \quad x, y \in G. \tag{1.5}$$

Proof. In the proof we use Gy. Maksa and Volkmann's [13] and Gilányi [9] results to prove that f^e is a solution of the quadratic functional equation (1.2) and f^o is a solution of the Cauchy functional equation.

Writing $x = y = 0$ in (1.4), we obtain $3\|f(0)\| \leq \|f(0)\|$, so $f(0) = 0$.
Replacing y by $-x$ in (1.4), we get

$$2f(x) + 2f^e(x) = f(2x). \tag{1.6}$$

By using $f = f^e + f^o$ and (1.6), we have

$$4f^e(x) + 2f^o(x) = f^e(2x) + f^o(2x). \tag{1.7}$$

If we replace x by $-x$ in (1.7), we get

$$4f^e(x) - 2f^o(x) = f^e(2x) - f^o(2x). \tag{1.8}$$

By adding and subtracting (1.7) to (1.8), we obtain respectively,

$$f^e(2x) = 4f^e(x) \tag{1.9}$$

and

$$f^o(2x) = 2f^o(x). \tag{1.10}$$

By using (1.9) and (1.10), we can easy to check by induction that

$$f^e(x) = 4^{-n}f^e(2^n x) \ \ \textit{and} \ \ f^o(x) = 2^{-n}f^o(2^n x) \tag{1.11}$$

for all $n \in \mathbb{N}$ and $x \in G$.
Substituting x by $-x$ and y by $-y$ in (1.4), we have

$$\|2f(-x) + 2f^e(y) - f(-x+y))\| \leq \|f(-x-y)\|, \quad x, y \in G. \tag{1.12}$$

By adding (1.12) to (1.4) and using the triangle inequality, we obtain

$$\|4f^e(x) + 4f^e(y) - 2f^e(x-y)\| \leq \|f(x+y)\| + \|f(-x-y)\|. \tag{1.13}$$

Writing $2^n x$ instead of x and $2^n y$ instead of y in (1.13), we get

$$\begin{aligned}\|4f^e(2^n x) + 4f^e(2^n y) - 2f^e(2^n(x-y))\| \leq \|f^e(2^n(x+y)) + f^o(2^n(x+y))\| \\ + \|f^e(2^n(x+y)) + f^o(2^n(-x-y))\|.\end{aligned} \tag{1.14}$$

By using the induction assumption (1.11) and dividing the new inequality by 4^n, we have

$$\|4f^e(x) + 4f^e(y) - 2f^e(x-y)\| \leq \|f^e(x+y) + 2^{-n}f^o(x-y)\| + \|f^e(x+y) + 2^{-n}f^o(-x-y)\|.$$

By letting $n \to +\infty$ in the last inequality, we obtain

$$\|2f^e(x) + 2f^e(y) - f^e(x-y)\| \leq \|f^e(x+y)\|, \quad x, y \in G. \tag{1.15}$$

It was proved in [9, 19] that this inequality is equivalent to the quadratic functional equation

$$f^e(x+y) + f^e(x-y) = 2f^e(x) + 2f^e(y), \quad x, y \in G. \tag{1.16}$$

Which proves the first part of our statement. From (1.4) we have

$$\|2f^e(x) + 2f^o(x) + 2f^e(y) - f^e(x-y) - f^o(x-y)\| \leq \|f(x+y)\|. \tag{1.17}$$

Since f^e satisfies the quadratic functional equation (1.16), we get

$$\|f^e(x+y) + 2f^o(x) - f^o(x-y)\| \leq \|f(x+y)\|. \tag{1.18}$$

Interchanging the roles of x and y in (1.18) we obtain

$$\|f^e(y+x) + 2f^o(y) - f^o(y-x)\| \leq \|f(y+x)\|. \tag{1.19}$$

Adding this inequality to (1.18), we get

$$\|f^e(x+y) + f^o(x) + f^o(y)\| \leq \|f^e(x+y) + f^o(x+y)\| \text{ for all } x, y \in G. \tag{1.20}$$

Inequality (1.20) can be rewritten as follows

$$\begin{aligned}&\|f^o(x) + f^o(y)\|^2 + \|f^e(x+y)\|^2 + 2Re\langle f^o(x) + f^o(y), f^e(x+y)\rangle \\ &\leq \|f^e(x+y)\|^2 + \|f^o(x+y)\|^2 + 2Re\langle f^o(x+y), f^e(x+y)\rangle, \quad \text{so,}\end{aligned}$$

$$\|f^o(x) + f^o(y)\|^2 + 2Re\langle f^o(x) + f^o(y) - f^o(x+y), f^e(x+y)\rangle \leq \|f^o(x+y)\|^2. \tag{1.21}$$

In (1.21), write $-x$ and $-y$ instead of x and y, respectively and add the inequality so obtained to (1.21) to obtain

$$\|f^o(x) + f^o(y)\| \leq \|f^o(x+y)\|, \quad x, y \in G. \tag{1.22}$$

In view of [13], the inequality (1.22) is equivalent to the Cauchy functional equation

$$f^o(x+y) = f^o(x) + f^o(y), \quad x, y \in G. \tag{1.23}$$

Thus, since $f(x) = f^e(x) + f^o(x)$, we can easily check that f is a solution of Drygas functional equation (1.5). This completes the proof. ♠ Q.E.D.

The commutativity of G used in Theorem 2.1 may be replaced by the Kannappan condition: $f(xyz) = f(yxz)$ for all $x, y, z \in G$.

Corollary 2.2. If G is group (not necessarily abelian) and E an inner product space. Then the Drygas inequality

$$\|2f(x) + f(y) + f(y^{-1}) - f(xy^{-1})\| \leq \|f(xy)\|$$

with $f(xyz) = f(yxz)$ for all $x, y, z \in G$, is equivalent to Drygas functional equation

$$f(xy) + f(xy^{-1}) = 2f(x) + f(y) + f(y^{-1}) \quad \text{for all } x, y \in G.$$

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